The harmonic gauge condition in the gravitomagnetic equations

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Abstract

It has been asserted in the literature that the analogy between the linear and first order slow motion approximation of Einstein equations of General Relativity (gravitomagnetic equations) and the Maxwell-Lorentz equations of electrodynamics breaks down if the gravitational potentials are time dependent. In this work, we show that this assertion is not correct and it has arisen from an incorrect limit of the usual harmonic gauge condition, which drastically changes the physical content of the gravitomagnetic equations.

1 Introduction

It has been asserted in [1] and also quoted in [2], that the analogy between the linear and first order slow motion approximations of Einstein equations of general relativity (GR) and Maxwell-Lorentz equations of electrodynamics, breaks down if the gravitational potentials are time dependent. This assertion comes, in our opinion, from the use of an incorrect limit of the usual harmonic gauge condition, which implies the absence of the gravitomagnetic induction term in the Faraday-like equation for the gravitoelectric field **g**. On the other hand, the presence of the gravitomagnetic induction term is essential in order to consider the possibility of existence of a spin dynamo. This possibility, in spite of the results exposed in [3], merits further study when the fluid equations are combined with the gravitomagnetic equations.

In this paper, we first review the linear approximation of GR in terms of the gravitational potentials and, in second place, in terms of a object in which only appear first

1

derivatives of the gravitational potentials. In both cases, the harmonic gauge condition is imposed. After, we shall show that from the latter equations, when also the first order slow motion approximation is considered and even when the gravitational potentials are time dependent, one obtains gravitomagnetic equations, which are completely analogous to the Maxwell-Lorentz equations of electrodynamics. The only difference is that the sign of the sources is reversed, to account for the attractive nature of gravity between masses.

2   Linear Einstein equations in terms of the gravitational potentials

The procedure for linearizing the geometrical left hand side part of Einstein's field equations for weak gravitational fields, is included in almost all texts on GR, see for instance [2,4]. Starting with Einstein field equations:

$$R_{ab} - \frac{1}{2}g_{ab}R = -\frac{8\pi G}{c^4}T_{ab} \tag{1}$$

We retain the universal constants G and c throughout this work, in order to know the different orders of the approximations. If the gravitational field is weak, then the metric tensor can be approximated by

$$g_{ab} \simeq \eta_{ab} + h_{ab} \tag{2}$$

where latin indexes $a, b = 0, 1, 2, 3$ and $\eta_{ab} = (+1, -1, -1, -1)$ is the flat Minkowski spacetime metric tensor, and h_{ab} is the gravitational perturbation to the flat metric. When gravity is weak, the linear approximation to GR should be valid. Thus, we now retain first order terms only, i.e., we neglect non-linear terms in h_{ab} and its derivatives in the Einstein equations, so R_{ab} and R are also correct to first order. The Ricci tensor can be calculated from the tensorial contraction of the Riemann tensor and the curvature scalar can be calculated from the contraction of the Ricci tensor and read

$$R_{ab} \simeq -\frac{1}{2}\Box h_{ab} \tag{3}$$

$$R \simeq \eta^{ab}R_{ab} = -\frac{1}{2}\Box h, \tag{4}$$

where $\Box$ is the D'Alembertian operator and to obtain (3) and (4) we have chosen our coordinate system as harmonic, so that we have chosen the following harmonic or de

2

Donder gauge condition (four components):

$$\left[h_{ab} - \frac{1}{2}\eta_{ab}h \right]_{,b} = 0. \tag{5}$$

A similar gauge condition, but with one scalar component, in Maxwell-Lorentz electrodynamics is named Lorenz (not from H. A. Lorentz). If we substitute the Ricci tensor (3) and the curvature scalar (4) into Einstein's equations, one obtains

$$-\frac{1}{2}\Box h_{ab} + \frac{1}{4}\eta_{ab}\Box h = -\frac{8\pi G}{c^4}T_{ab} \tag{6}$$

We now define the gravitational potentials as

$$\overline{h}_{ab} = h_{ab} - \frac{1}{2}\eta_{ab}h \tag{7}$$

and substitute equation (7) into equation (6) and rearrange, one gets

$$\Box \overline{h}_{ab} = \frac{16\pi G}{c^4}T_{ab} \tag{8}$$

If we write out the D'Alembertian operator, we have

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\overline{h}_{ab} - \triangle \overline{h}_{ab} = \frac{16\pi G}{c^4}T_{ab} \tag{9}$$

This is the basic equation in terms of the gravitational potential, upon which all the analogies between electromagnetism and gravity are based, and leads to some predictions, as for instance, the existence of linear gravitational waves.

On the other hand, the 3+1 spacetime splitting, see [5], regards three dimensional space as curved rather than Euclidean and its metric g_{ik} is the spatial part of the spacetime metric g_{ab}. In this curved three space reside two gravitational potentials: a "gravitoelectric" Newtonian scalar potential Φ, which resides in the time-time part g_{00} of the space-time metric; and a "gravitomagnetic" vector potential $\mathbf{a}$, which is essentially the time-space part g_{0j} of the space-time metric. The decomposition of g_{ab} into g_{ik}, Φ and $\mathbf{a}$ is analogous to the decomposition of the four vector electromagnetic potential A_a into an electric scalar potential $\Psi = A_0$ and a magnetic vector potential $\mathbf{A} = A_j$.

3 Gravitomagnetic equations in terms of fields

The analogy with Maxwell-Lorentz electrodynamics can be further continued (see [6,3]), by writing the linear gravitational equations in terms of first derivatives of the gravitational potential, i.e., acceleration fields. For doing this, we first introduce the object

$$G^{abc} = \frac{1}{4}\left(\overline{h}^{ab,c} - \overline{h}^{ac,b} + \eta^{ab}\,\overline{h}^{cd}{}_{,d} - \eta^{ac}\,\overline{h}^{bd}{}_{,d}\right) \tag{10}$$

Impose now the harmonic de Donder gauge condition:

$$\overline{h}^{ab}{}_{,b} = 0. \tag{11}$$

From (10) and (11) one obtains:

$$G^{abc} = \frac{1}{4}\left(\overline{h}^{ab,c} - \overline{h}^{ac,b}\right) \tag{12}$$

Its properties are:

$$G^{a[bc]} = G^{abc}, \tag{13}$$

$$G^{[abc]} = 0, \tag{14}$$

$$G^{d[ab,c]} = 0, \tag{15}$$

where $[\cdot\cdot]$ is the antisymmetry symbol. Taking (2),(10) and (11) into (1) and keeping only linear terms, one obtains the weak field equations in terms of the object G^{abc}, in which only first derivatives of the gravitational potential appear:

$$\frac{\partial G^{abc}}{\partial x^c} = -\frac{4\pi G}{c^4}T^{ab}. \tag{16}$$

Introducing the gravitoelectric newtonian scalar potential Φ and the gravitomagnetic vector potential $\mathbf{a}$ as

$$\Phi = -\frac{c^2\overline{h}^{00}}{4} \tag{17}$$

$$a^i = \frac{c^2\overline{h}^{0i}}{4} \qquad \mathbf{a} = \left(a^1, a^2, a^3\right). \tag{18}$$

Introduce new symbols, and substitute equations (17) and (18) into equation (12), to get the gravitoelectric field $\mathbf{g}$ as

$$G^{00i} = \frac{1}{4}\left(\overline{h}^{00,i} - \overline{h}^{0i,0}\right), \tag{19}$$

4

$$g^i = c^2 G^{00i} = -\frac{\partial \Phi}{\partial x^i} - \frac{1}{c}\frac{\partial a^i}{\partial t}, \tag{20}$$

$$\mathbf{g} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{a}}{\partial t}, \tag{21}$$

and the gravitomagnetic field $\mathbf{b}$ as

$$\mathbf{b} = (b^1, b^2, b^3), \quad b^1 = c^2 G^{023}, b^2 = c^2 G^{031}, b^3 = c^2 G^{012} \tag{22}$$

$$\mathbf{b} = \nabla \wedge \mathbf{a} \qquad\qquad c^2 G^{0ij} = a^{i,j} - a^{j,i}.$$

On the other hand, performing now the first order slow motion approximation for the energy-momentum tensor, we assume that the source masses involved have appreciable velocity $\mathbf{v}$ or rotation, but neglect quadratic terms in velocity, i.e., neglect the stress part of the energy-momentum tensor. Thus, the energy-momentum tensor will only have the components

$$T^{00} = \rho c^2 \tag{23}$$

and

$$T^{0i} = \rho c v^i. \tag{24}$$

Put (20), (22) (23) and (24) into (16) and (13), one obtains, when the first order effects of the motion of the sources are taken into account, the following *gravitomagnetic* (Mawwell-like) equations:

$$\nabla\mathbf{g} = -4\pi G\rho, \tag{25}$$

$$\nabla \wedge \mathbf{b} = -\frac{4\pi G}{c}\rho\,\mathbf{v} + \frac{1}{c}\frac{\partial \mathbf{g}}{\partial t}, \tag{26}$$

$$\nabla \wedge \mathbf{g} = -\frac{\partial \mathbf{b}}{\partial t}, \tag{27}$$

$$\nabla\mathbf{b} = 0. \tag{28}$$

From the four harmonic gauge conditions one obtains that, at order c^{-2}, the potentials verify only the Lorenz-like one:

$$\nabla\mathbf{a} + \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0. \tag{29}$$

At this point it must be emphasized that in [1], the introduction of an additional vector gauge condition was considered, which reads

$$\frac{\partial \mathbf{a}}{\partial t} = 0, \tag{30}$$

5

as also coming from the harmonic gauge conditions. This is in our opinion incorrect, and moreover, causes the disappearance of the induction term in eq.(27).

However, in the particular case in which the weak gravity field is stationary, i.e., if the gravitational potentials are independent of time

$$\overline{h}^{00}{}_{,0} = 0, \qquad \overline{h}^{0i}{}_{,0} = 0, \tag{31}$$

or equivalently, due to (17) and (18),

$$\frac{\partial \Phi}{\partial t} = 0, \qquad \frac{\partial \mathbf{a}}{\partial t} = 0, \tag{32}$$

then the gravitomagnetic equations are transformed into the following ones

$$\nabla \mathbf{g} = -4\pi G \rho, \tag{33}$$

$$\nabla \wedge \mathbf{b} = -\frac{4\pi G}{c} \rho \mathbf{v}, \tag{34}$$

$$\nabla \wedge \mathbf{g} = 0, \tag{35}$$

$$\nabla \mathbf{b} = 0. \tag{36}$$

and in this case the gauge condition is the Coulomb-like one

$$\nabla \mathbf{a} = 0, \tag{37}$$

where now $\mathbf{g}$ is the Newtonian acceleration field

$$\mathbf{g} = -\nabla \Phi. \tag{38}$$

Finally, in the static case ($\overline{h}^{0i} = 0$), the gravitomagnetic field $\mathbf{b}$ is zero and one obtains the Newtonian field equations:

$$\nabla \mathbf{g} = -4\pi G \rho, \tag{39}$$

$$\nabla \wedge \mathbf{g} = 0. \tag{40}$$

Also, for a weak stationary gravity field, from the geodesic equation (equations of motion)

$$\frac{d^2 x^a}{d\tau^2} + \Gamma^a_{bc} \frac{dx^b}{d\tau} \frac{dx^c}{d\tau} = 0, \tag{41}$$

one obtains the Lorentz-like force law, which reads

$$\frac{d\mathbf{u}}{dt} = \mathbf{g} + \frac{4}{c} \mathbf{u} \wedge \mathbf{b}, \tag{42}$$

6

where $\mathbf{u}$ is the velocity of the test particle. Note the factor 4 in the gravitomagnetic force term, which signals the difference with respect to electrodynamics. Moreover, for a weak stationary field one obtains the gravitomagnetic potential

$$\mathbf{a} = -\frac{1}{2}\frac{\mathbf{S} \wedge \mathbf{r}}{cr^3}, \tag{43}$$

and the gravitomagnetic field

$$\mathbf{b} = \nabla \wedge \mathbf{a} = -\frac{1}{2}\frac{3\mathbf{n}(\mathbf{S} \cdot \mathbf{n}) - \mathbf{S}}{cr^3}, \tag{44}$$

where $\mathbf{S}$ is the intrinsic angular momentum of the source and $\mathbf{n}$ is the unit position vector. These equations are analogous to the electromagnetic ones, changing the magnetic dipole moment by minus half the angular momentum. They are used in the GP-B gyroscope and LAGEOS III experiments (see [7]), to obtain the precession of test bodies due to stationary gravitomagnetic $\mathbf{b}$ field generated by the rotation of the Earth mass.

4 Conclusion

The gravitational potentials Φ and $\mathbf{a}$ can be time dependent, for instance, due to changes in time of the mass, intrinsic angular momentum or distance of(to) the source. In this work, we have showed that the gravitomagnetic equations, with an induction term in the Faraday-like one, must be applied in this case, when the linear and first order slow motions approximations of the gravitational field are considered.

5 ACKNOWLEDGEMENTS

6   REFERENCES

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